Petrov-Galerkin method with cubic B-splines for solving the MEW equation

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Abstract

In the present paper, we introduce a numerical solution algorithm based on a Petrov-Galerkin method in which the element shape functions are cubic B-splines and the weight functions quadratic B-splines . The motion of a single solitary wave and interaction of two solitary waves are studied. Accuracy and efficiency of the proposed method are discussed by computing the numerical conserved laws and L_2 , L_∞ error norms. The obtained results show that the present method is a remarkably successful numerical technique for solving the modified equal width wave(MEW) equation. A linear stability analysis of the scheme shows that it is unconditionally stable.

1 Introduction

The present study is concerned with one-dimensional modified equal width wave (MEW) equation

$$U_t + 3U^2U_x - \mu U_{xxt} = 0, \tag{1}$$

with the physical boundary conditions $U \to 0$ as $x \to \pm\infty$, where t is time, x is the space coordinate, μ is a positive parameter and $U(x,t)$ is wave amplitude. The MEW equation, which we discuss here, is related with the modified regularized long wave (MRLW)equation [1] and modified Korteweg-de Vries (MKdV) equation [2] and is based upon the equal width wave (EW) equation. This equation has solitary wave solutions with both positive and negative amplitudes, all

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of which have the same width. The MEW equation is a non-linear wave equation with cubic nonlinearity with a pulse-like solitary wave solution [3]. The MEW equation with a limited set of boundary and initial conditions has an analytical solutions. Therefore numerical solutions of the MEW equation have been the subject of many papers. Zaki considered the solitary wave interactions for the MEW equation by Collocation method using quintic B-spline finite elements [4] and obtained the numerical solution of the EW equation by using the least-squares method [5]. Variational iteration method is introduced to solve the MEW equation by Junfeng Lu [6]. Wazwaz investigated the MEW equation and two of its variants by the tanh and the sine-cosine methods [3]. Esen applied a lumped Galerkin method based on quadratic B-spline finite elements have been used for solving the EW and MEW equation[7, 8]. Saka proposed algorithms for the numerical solution of the MEW equation using quintic B-spline collocation method [9]. A. Esen and S. Kutluay studied a linearized implicit finite difference method in solving the MEW equation [10]. D. J. Evans, and K. R. Raslan [11] studied the generalized equal width (GEW) equation by using collocation method based on quadratic B-splines to obtain the numerical solutions of a single solitary waves, and the birth of solitons. K. R. Raslan [12] obtained the numerical solutions of the GEW equation by collocation method using cubic B-spline. T. Geyikli and S. Battal Gazi Karakoç [13, 14, 15] obtained the numerical solution of the MEW equation with septic B-spline collocation method, a lumped Galerkin method based on cubic B-spline finite element method and subdomain method using quartic B-spline functions. L. R. T. Gardner, G. A. Gardner and T. Geyikli [16] solved the KdV equation numerically by a Petrov-Galerkin method and results are almost nearly equal to exact solutions. Abdulkadir Doğan[17] studied the RLW equation numerically using the Petrov-Galerkin method and obtained accurate results. In this paper, we have applied a lumped Petrov-Galerkin method in which the element shape functions are cubic B-splines and the weight functions are quadratic B-splines. The performance and accuracy of the proposed method have been tested on two problems: the motion of a single solitary wave and the interaction of two solitary waves. A linear stability analysis based on a Fourier method shows that the numerical scheme is unconditionally stable.

2 Cubic B-spline Petrov-Galerkin method

For the numerical treatment, the solution domain of the problem is restricted over an interval $a \leq x \leq b$. Consider the equation (1) with the boundary conditions

$$\begin{aligned} &U(a,t)=0, \quad U(b,t)=0, \\ &U_x(a,t)=0, \;\; U_x(b,t)=0, \;\; t>0, \end{aligned} \tag{2}$$

and the initial condition

$$U(x,0)=f(x), \quad a \leq x \leq b$$

where $f(x)$ is a prescribed function. Physical boundary conditions require U and $U_x \rightarrow 0$ that $U \rightarrow 0$ for $x \rightarrow \pm\infty$. The finite interval $[a,b]$ is partitioned into N finite elements of equal length h by the nodes x_m such that $a = x_0 < x_1 \cdots < x_N = b$ and $h = (x_{m+1} - x_m)$. The cubic B-splines $\phi_m(x)$, (m= -1(1) N+1), at the knots x_m which form a basis over the interval $[a, b]$ are defined by the relationships [18]

$$\phi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}], \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 & \\ \qquad -3(x - x_{m-1})^3 & x \in [x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 & \\ \qquad -3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}], \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}], \\ 0 & \textit{otherwise}. \end{cases} \tag{3}$$

The numerical solution $U_N(x, t)$ is expressed in terms of the cubic B-splines and unknown time dependent parameters as

$$U_N(x, t) = \sum_{j=-1}^{N+1} \delta_j(t)\phi_j(x) \tag{4}$$

where δ_j are time dependent quantities to be determined from the boundary and weighted residual conditions. Each cubic B-spline covers 4 elements so that each element $[x_m, x_{m+1}]$ is covered by 4 splines. In terms of a local coordinate system η given by

$$h\eta = x - x_m \quad 0 \leq \eta \leq 1 \tag{5}$$

so the cubic B-spline shape functions over the element $[x_m, x_{m+1}]$ can be defined as

$$\begin{aligned} \phi_{m-1} &= (1-\eta)^3, \\ \phi_m &= 1 + 3(1-\eta) + 3(1-\eta)^2 - 3(1-\eta)^3, \\ \phi_{m+1} &= 1 + 3\eta + 3\eta^2 - 3\eta^3, \\ \phi_{m+2} &= \eta^3. \end{aligned} \tag{6}$$

All splines apart from $\phi_{m-1}(x), \phi_m(x), \phi_{m+1}(x)$ and $\phi_{m+2}(x)$ are zero over the element $[x_m, x_{m+1}]$. Over the typical element $[x_m, x_{m+1}]$, the numerical solution $U_N(x, t)$ is given by

$$U_N(x, t) = \sum_{j=m-1}^{m+2} \phi_j(x)\delta_j(t) \tag{7}$$

where $\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}$ act as element parameters and B-splines ϕ_{m-1}, ϕ_m, ϕ_{m+1}, ϕ_{m+2} as element shape functions. Using trial function (4) and cubic splines (3), the values of U, U', U'' at the knots are determined in terms of the element parameters δ_m by

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1}, \\ U'_m &= U'(x_m) = \tfrac{3}{h}(-\delta_{m-1} + \delta_{m+1}), \\ U''_m &= U''(x_m) = \tfrac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}), \end{aligned} \tag{8}$$

where the symbols $'$ and $''$ denotes first and second differentiation with respect to x, respectively. The splines $\phi_m(x)$ and its two principle derivatives vanish outside the interval $[x_{m-2}, x_{m+2}]$. The weight function $W(x)$ is taken a quadratic B-spline Ψ_m. Quadratic B-spline Ψ_m at the knots x_m are defined over the interval $[a, b]$ by

$$\Psi_m(x) = \frac{1}{h^2}\begin{cases} (x_{m+2}-x)^2 - 3(x_{m+1}-x)^2 + 3(x_m-x)^2, & [x_{m-1}, x_m] \\ (x_{m+2}-x)^2 - 3(x_{m+1}-x)^2 & [x_m, x_{m+1}] \\ (x_{m+2}-x)^2 & [x_{m+1}, x_{m+2}] \\ 0 & otherwise. \end{cases} \tag{9}$$

Using the local coordinate transformation (5) for the finite element $[x_m, x_{m+1}]$ quadratic B-spline shape functions can be defined as

$$\begin{aligned} \Psi_{m-1} &= (1-\eta)^2, \\ \Psi_m &= 1 + 2\eta - 2\eta^2, \\ \Psi_{m+1} &= \eta^2. \end{aligned} \tag{10}$$

When the Petrov-Galerkin approach is applied to Eq.(1), we obtain the weak form of (1)

$$\int_a^b W(U_t + 3U^2U_x - \mu U_{xxt} dx) = 0. \tag{11}$$

Using transformation (5), equation(11) for the typical element $[x_m, x_{m+1}]$ becomes

$$\int_0^1 W\left(U_t + \frac{3}{h}\hat{U}^2U_\eta - \frac{\mu}{h^2}U_{\eta\eta t}\right) d\eta = 0 \tag{12}$$

where $\hat{U}$ is taken to be a constant over the element $[x_m, x_{m+1}]$ to simplify the integral. Integrating equation (12) by parts and using Eq.(1)leads to

$$\int_0^1 [W(U_t + \lambda U_\eta) + \beta W_\eta U_{\eta t}\,] d\eta = \beta W U_{\eta t}|_0^1 \tag{13}$$

where $\lambda = \frac{3\hat{U}^2}{h}$ and $\beta = \frac{\mu}{h^2}$. Taking the weight function $W(x) = \Psi_m$ with quadratic B-spline shape functions given by equation (10) and substituting approximation (7)into integral equation (13), we obtain the element contributions in the form

$$\sum_{j=m-1}^{m+2} [(\int_0^1 \Psi_i\phi_j + \beta\Psi_i'\phi_j')d\eta - \beta\Psi_i\phi_j'|_0^1\,]\dot{\delta}_j^e + \sum_{j=m-1}^{m+2}(\lambda\int_0^1 \Psi_i\phi_j' d\eta)\delta_j^e \tag{14}$$

which can be written in matrix form as follows:

$$[A^e + \beta(B^e - C^e)]\dot{\delta}^e + \lambda D^e\delta^e \tag{15}$$

where $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$ are the element parameters and the dot denotes differentiation with respect to t. The element matrices A^e, B^e, C^e and D^e are rectangular 3×4 given by the following integrals:

$$A_{ij}^e = \int_0^1 \Psi_i\phi_j d\eta = \frac{1}{60}\begin{bmatrix} 10 & 71 & 38 & 1 \\ 19 & 221 & 221 & 19 \\ 1 & 38 & 71 & 10 \end{bmatrix}$$

$$B^e_{ij} = \int_0^1 \Psi'_i \phi'_j d\eta = \frac{1}{2}\begin{bmatrix} 3 & 5 & -7 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & -7 & 5 & 3 \end{bmatrix}$$

$$C^e_{ij} = \Psi_i \phi'_j|_0^1 = \frac{3}{1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$D^e_{ij} = \int_0^1 \Psi_i \phi'_j d\eta = \frac{1}{10}\begin{bmatrix} -6 & -7 & 12 & 1 \\ -13 & -41 & 41 & 13 \\ -1 & -12 & 7 & 6 \end{bmatrix}$$

where i takes only the values $1, 2, 3$ and the j takes only the values $m-1, m, m+1, m+2$ for the typical element $[x_m, x_{m+1}]$. A lumped value for λ is found from $\frac{1}{4}(U_m + U_{m+1})^2$ as

$$\lambda = \frac{3}{4h}(\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^2.$$

Assembling all contributions from all elements leads to the following matrix equation

$$[A + \beta(B - C)]\dot{\delta} + \lambda D\delta = 0 \tag{16}$$

where $\delta = (\delta_{-1}, \delta_0, ..., \delta_N, \delta_{N+1})^T$ is a global element parameters. The matrices A, B and λD are rectangular and row m of each has the following form:

$A = \frac{1}{60}(1, 57, 302, 302, 57, 1, 0),$

$B = \frac{1}{2}(-1, -9, 10, 10, -9, -1, 0),$

$\lambda D = \frac{1}{10}(-\lambda_1, -12\lambda_1 - 13\lambda_2, 7\lambda_1 - 41\lambda_2 - 6\lambda_3, 6\lambda_1 + 41\lambda_2 - 7\lambda_3,$
$13\lambda_2 + 12\lambda_3, \lambda_3, 0)$

where

$\lambda_1 = \frac{3}{4h}(\delta_{m-2} + 5\delta_{m-1} + 5\delta_m + \delta_{m+1})^2,\ \lambda_2 = \frac{3}{4h}(\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^2,$
$\lambda_3 = \frac{3}{4h}(\delta_m + 5\delta_{m+1} + 5\delta_{m+2} + \delta_{m+3})^2.$

Using the Crank-Nicholson approach $\delta = \frac{1}{2}(\delta^n + \delta^{n+1})$ and the forward finite difference $\dot{\delta} = \frac{\delta^{n+1} - \delta^n}{\Delta t}$ in equation (16), we obtain the following $(N+2) \times (N+3)$ matrix system

$$[A + \beta(B - C) + \frac{\lambda \Delta t}{2} D]\delta^{n+1} = [A + \beta(B - C) - \frac{\lambda \Delta t}{2} D]\delta^n \tag{17}$$

where Δt is the time step. Applying the boundary conditions (2) to the system(17) we make the matrix equation square. The resulting matrices are asymmetrically banded but may be taken depleted septadiagonal so are efficiently solved with a variant of the Thomas algorithm. Three or four inner iterations are applied to $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ at each time in order to improve the accuracy. The initial vector δ^0 is determined from the initial and boundary conditions. So the approximation (7) can be rewritten for the initial condition

$$U_N(x,0) = \sum_{m=-1}^{N+1} \phi_m(x)\delta_m^0, \tag{18}$$

where the parameters δ_m^0 will be determined. Using relations at the knots $U_N(x_m,0) = U(x_m,0)$, $m = 0,1,\cdots,N$, together with derivative condition $U_N'(x_0,0) = U'(x_N,0) = 0$, the initial vector δ^0 can be determined from the following matrix equation

$$\begin{bmatrix} -3 & 0 & 3 & & & \\ 1 & 4 & 1 & & & \\ & & \ddots & & & \\ & & & 1 & 4 & 1 \\ & & & -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} \delta_{-1}^0 \\ \delta_0^0 \\ \vdots \\ \delta_N^0 \\ \delta_{N+1}^0 \end{bmatrix} = \begin{bmatrix} 0 \\ U(x_0) \\ \vdots \\ U(x_N) \\ 0 \end{bmatrix}$$

which can be solved using a variant of the Thomas algorithm.

2.1 Stability analysis

A typical member of the matrix system (17) can be written in terms of the nodal parameters δ_m^n as

$$\begin{aligned} &\gamma_1\delta_{m-2}^{n+1} + \gamma_2\delta_{m-1}^{n+1} + \gamma_3\delta_m^{n+1} + \gamma_4\delta_{m+1}^{n+1} + \gamma_5\delta_{m+2}^{n+1} + \gamma_6\delta_{m+3}^{n+1} = \\ &\gamma_6\delta_{m-2}^{n} + \gamma_5\delta_{m-1}^{n+} + \gamma_4\delta_m^{n} + \gamma_3\delta_{m+1}^{n} + \gamma_2\delta_{m+2}^{n} + \gamma_1\delta_{m+3}^{n} \end{aligned} \tag{19}$$

where

$$\begin{aligned} \gamma_1 &= \tfrac{1}{60} - \tfrac{\beta}{2} - \tfrac{\lambda\Delta t}{20}, \\ \gamma_2 &= \tfrac{57}{60} - \tfrac{9\beta}{2} - \tfrac{25\lambda\Delta t}{20}, \\ \gamma_3 &= \tfrac{302}{60} + \tfrac{10\beta}{2} - \tfrac{40\lambda\Delta t}{20}, \\ \gamma_4 &= \tfrac{302}{60} + \tfrac{10\beta}{2} + \tfrac{40\lambda\Delta t}{20}, \\ \gamma_5 &= \tfrac{57}{60} - \tfrac{9\beta}{2} + \tfrac{25\lambda\Delta t}{20}, \\ \gamma_6 &= \tfrac{1}{60} - \tfrac{\beta}{2} + \tfrac{\lambda\Delta t}{20}. \end{aligned}$$

For the stability analysis it is convenient to use the Fourier method. To apply Fourier stability analysis, the MEW equation needs to be linearized by assuming that the quantity U in the non-linear term U^2U_x is locally constant. Substituting the fourier mode $\delta_j^n = \xi^n e^{ijkh}$ where k is mode number and h the element size into the scheme (19) we have

$$g = \frac{a - ib}{a + ib}, \tag{20}$$

where

$$\begin{aligned} a &= (302 + 300\beta)\cos(\tfrac{\theta}{2})h + (57 - 270\beta)\cos(\tfrac{3\theta}{2})h + (1 - 30\beta)\cos(\tfrac{5\theta}{2}), \\ b &= 120\lambda\Delta t\sin(\tfrac{\theta}{2})h + 75\lambda\Delta t\sin(\tfrac{3\theta}{2})h + 3\lambda\Delta t\sin(\tfrac{5\theta}{2})h. \end{aligned} \tag{21}$$

Taking the modulus of equation(20) , we have $|g| = 1$. Therefore the scheme is unconditionally stable.

3 Numerical examples and results

The purpose of this section is to examine the deduced algorithm using different test problems concerned with the motion of single solitary wave and interaction of two solitary waves. We use the L_2 and L_∞ error norms which are used to show how good the numerical results in comparison with the exact results defined by

$$L_2 = \left\|U^{exact} - U_N\right\|_2 \simeq \sqrt{h\sum_{J=0}^{N}\left|U_j^{exact} - (U_N)_j\right|^2}, \tag{22}$$

and the L_∞ error norm

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|. \tag{23}$$

For the MEW equation, we have calculated following invariants [19]

$$\begin{aligned} C_1 &= \textstyle\int_a^b U dx \simeq h\sum_{J=1}^{N} U_j^n, \\ C_2 &= \textstyle\int_a^b U^2 + \mu (U_x)^2 dx \simeq h\sum_{J=1}^{N} (U_j^n)^2 + \mu\,(U_x)_{j,}^n \\ C_3 &= \textstyle\int_a^b U^4 dx \simeq h\sum_{J=1}^{N} (U_j^n)^4. \end{aligned}$$

which correspond to conversation of mass, momentum and energy, respectively.

3.1 The motion of single solitary wave

An exact solution of this problem is given by [4]

$$U(x,t) = A\sec h(k[x - x_0 - vt])$$

where the wave velocity $v = \frac{A^2}{2}$ and $k = \sqrt{\frac{1}{\mu}}$. This equation represents a single solitary wave of amplitude A, initially centered on x_0. The initial condition is taken as

$$U(x,0) = A\sec h(k[x - x_0]).$$

with $x_0 = 30$ and boundary condition $U \to 0$ as $x \to \pm\infty$. For this problem the analytical values of the invariants are [4]

$$C_1 = \frac{A\pi}{k}, \; C_2 = \frac{2A^2}{k} + \frac{2\mu k A^2}{3}, \; C_3 = \frac{4A^4}{3k}. \tag{24}$$

This problem was solved on the interval $0 \le x \le 80$ with the parameters $h = 0.1$, $\Delta t = 0.05$, $\mu = 1, x_0 = 30, A = 0.25$. The analytical values of invariants are obtained from equation (24) $C_1 = 0.7853982, C_2 = 0.1666667, C_3 = 0.0052083$. The computations are done until time $t = 20$ to find error norms L_2 , L_∞ and numerical invariants C_1, C_2, C_3 at various times. Changes of invariants C_1 and C_2 are extremely small and less than 1×10^{-7} and 2×10^{-7} percent, respectively, and C_3 remain constant during the computer run. Results are documented in Table 1.

We can easily see that the results of numerical scheme implies the highest accuracy and the obtained results are compared with some previous results. Figure 1 shows that the proposed method perform the motion of propagation of a solitary wave satisfactorily, which moves to the right at a constant speed and preserves its amplitude and shape with increasing time as expected. Amplitude is 0.25 at $t = 0$ which is located at $x = 30$, while it is 0.249900 at $t = 20$ which is located at $x = 30.6$. The absolute difference in amplitudes at times $t = 0$ and $t = 20$ is 1×10^{-4} so that there is a little change between amplitudes.

Table 1: *Invariants and error norms for single solitary wave with* $h = 0.1, \Delta t = 0.05, A = 0.25, (0 \leq x \leq 80)$.

t	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	0.7853966	0.1666661	0.0052083	0.0000000	0.0000000
5	0.7853966	0.1666662	0.0052083	0.0204940	0.0115654
10	0.7853966	0.1666662	0.0052083	0.0407954	0.0231931
15	0.7853967	0.1666662	0.0052083	0.0607312	0.0347680
20	0.7853967	0.1666663	0.0052083	0.0801465	0.0461218
20[8]	0.7853898	0.1667614	0.0052082	0.0796940	0.0465523
20[10]	0.7853977	0.1664735	0.0052083	0.2692812	0.2569972
20[11]	0.7849545	0.1664765	0.0051995	0.2905166	0.2498925
20[12]	0.7844667	0.1664340	0.0051938	0.1958878	0.1744330

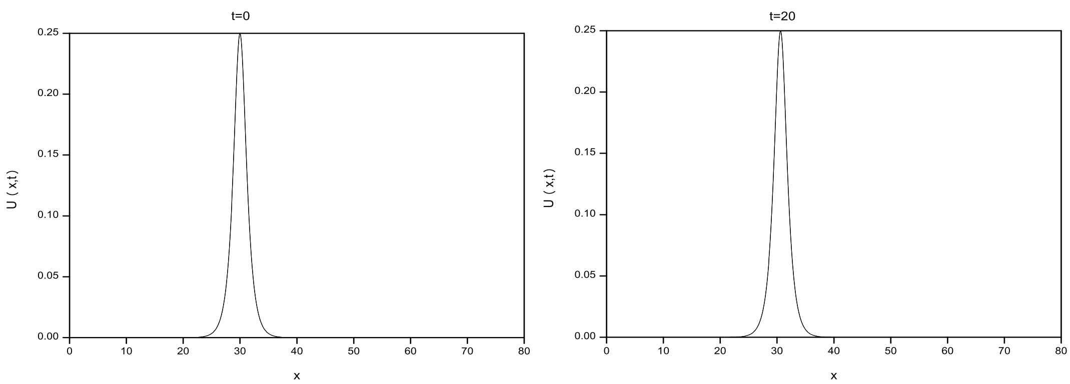

Figure 1: *The motion of a single solitary wave with* $h = 0.1, \Delta t = 0.05$ *at* $t = 0$ *and* $t = 20$.

We compute the convergence rates for the numerical method in space sizes h_m and time steps $U_{\Delta t_m}$ with the following formula [8]:

$$order = \frac{log_{10}(|U^{exact} - U^{num}_{h_m}|/|U^{exact} - U^{num}_{h_m+1}|)}{log_{10}(h_m/h_{m+1})}$$

and

$$order = \frac{log_{10}(|U^{exact} - U^{num}_{\Delta t_m}|/|U^{exact} - U^{num}_{\Delta t_m+1}|)}{log_{10}(\Delta t_m/\Delta t_{m+1})}$$

The computed convergence rates for values of space size h_m and a fixed value of the time step Δt_m are given in Table 2. We have clearly seen that the convergence rates when Δt is fixed are not as good as for the space sizes. In addition, the time rate of the convergence for the numerical method is computed with various time step $U_{\Delta t}$ and fixed space step h in Table 3. It can be seen that the present method provides remarkable reductions in convergence rates for the smaller time.

Table 2: *The order of convergence at* $t = 20, \Delta t = 0.05, A = 0.25, (0 \leq x \leq 80)$.

h_m	$L_2 \times 10^3$	order	$L_\infty \times 10^3$	order
0.8	55.25579280	-	20.73190012	-
0.4	1.18988053	5.53723708	0.66333732	4.96596584
0.2	0.31651483	1.91047172	0.18136337	1.87736182
0.1	0.08014654	1.98156095	0.04612186	1.97536050
0.05	0.01944857	2.04446052	0.01131649	2.02702414
0.025	0.00542320	1.84244780	0.00312864	1.80853182

Table 3: *The order of convergence at* $t = 20, h = 0.1, A = 0.25, (0 \leq x \leq 80)$.

Δt_m	$L_2 \times 10^3$	order	$L_\infty \times 10^3$	order
0 .8	0.07690387	-	0.05102567	-
0.4	0.07901129	-0.03900261	0.04497304	0.18216281
0.2	0.07986051	-0.01542346	0.04582559	-0.02709304
0.1	0.08008880	-0.00411821	0.04606209	-0.00742641
0.05	0.08014654	-0.00103973	0.04612186	-0.00187082
0.025	0.08016098	-0.00025990	0.04613687	-0.00046943

3.2 Interaction of two solitary waves

In this section, we consider the MEW equation with boundary conditions $U \to 0$ as $x \to \pm\infty$ and the initial condition

$$U(x,0) = \sum_{j=1}^{2} A_j \sec h(k[x - x_j]),$$

where $k = \sqrt{\frac{1}{\mu}}$ which corresponds to two solitary waves. In our computational work, we first choose the parameters $h = 0.1$, $\mu = 1$, $\Delta t = 0.025, A_1 = 1, x_1 = 15, A_2 = 0.5, x_2 = 30$ over the interval $0 \leq x \leq 80$ used the earlier papers [4, 8, 9]. The analytic invariants are $C_1 = 4.7123889, C_2 = 3.3333333, C_3 = 1.4166667$ [11]. The experiment was run from $t = 0$ to $t = 55$ to allow the interaction in order to take place. The absolute difference between the values of the invariants obtained by the present method at times $t = 0$ and $t = 55$ are $\Delta C_1 = 2.77 \times 10^{-4}, \Delta C_2 = 4.21 \times 10^{-4}, \Delta C_3 = 1.53 \times 10^{-4}$. Figure 2 shows the interactions of two positive solitary waves. As it is seen from the figure 2, at $t = 5$ the wave with larger amplitude is on the left of the second wave with smaller amplitude. The larger wave catches up with the smaller one as time increases. Interaction started at about time $t = 25$, overlapping processes occurred between times $t = 25$ and $t = 40$ and waves started to resume their original shapes after time $t = 40$. At $t = 55$, the amplitude of larger waves is 0.999487 at the point $x = 44.4$ whereas the amplitude of the smaller one is 0.510461 at the point $x = 34.7$. It is found that the absolute difference in amplitude is 1.04×10^{-2} for the smaller wave and 0.513×10^{-3} for the larger wave for this algorithm.

Table 4: *Invariants for the interaction of two solitary waves* $h = 0.1, \Delta t = 0.025$.

	$A_1 = 1, A_2 = 0.5, (0 \leq x \leq 80)$			$A_1 = -2, A_2 = 1, (0 \leq x \leq 150)$		
t	C_1	C_2	C_3	C_1	C_2	C_3
0	4.7123732	3.3333253	1.4166643	-3.1415737	13.3332816	22.6665313
5	4.7123772	3.3333364	1.4166734	-3.1415444	13.3332606	22.6663984
10	4.7123784	3.3333387	1.4166747	-3.1296528	13.2813648	22.4920480
15	4.7123798	3.3333433	1.4166789	-3.1429737	13.3294715	22.6646333
20	4.7123874	3.3333684	1.4167043	-3.1418189	13.3331502	22.6675394
25	4.7124336	3.3335159	1.4168581	-3.1416556	13.3334961	22.6679597
30	4.7126472	3.3342279	1.4176446	-3.1416828	13.3335695	22.6681547
35	4.7128480	3.3350277	1.4186862	-3.1417274	13.3336177	22.6683287
40	4.7123922	3.3335242	1.4170670	-3.1417743	13.3336616	22.6684994
45	4.7122290	3.3330996	1.4166684	-3.1418214	13.3337043	22.6686696
50	4.7121724	3.3329959	1.4166151	-3.1418686	13.3337466	22.6688398
55	4.7121740	3.3330073	1.4166133	-3.1419159	13.3337885	22.6690099

In addition, we have chosen for the computational work $\mu = 1, x_1 = 15, x_2 = 30, A_1 = -2, A_2 = 1$ together with time step $\Delta t = 0.025$ and space step $h = 0.1$ in the range $0 \leq x \leq 150$. The experiment was run from $t = 0$ to $t = 55$ to allow the interaction to take place. Figure 3 shows the development of the solitary wave interaction. As is seen from the figure 3, at $t = 0$ a wave with negative amplitude is on the left of another wave with positive amplitude. The larger wave with the negative amplitude catches up with the smaller one with the positive amplitude as the time increases. At $t = 55$, the amplitude of the smaller wave is 0.974419 at the point $x = 52.5$, whereas the amplitude of the larger one is -1.988078 at the point $x = 122.8$. It is found that the absolute difference in amplitudes is 2.55×10^{-2} for the smaller wave and 1.19×10^{-1} for the larger wave. The analytical invariants can be found as $C_1 = -3.1415927, C_2 = 13.3333333, C_3 = 22.6666667$. Table 4 lists the values of the invariants of the two solitary waves with amplitudes $A_1 = 1, A_2 = 0.5$ and $A_1 = -2, A_2 = 1$. It is observed that the obtained values of the invariants remain almost constant during the computer run. These values

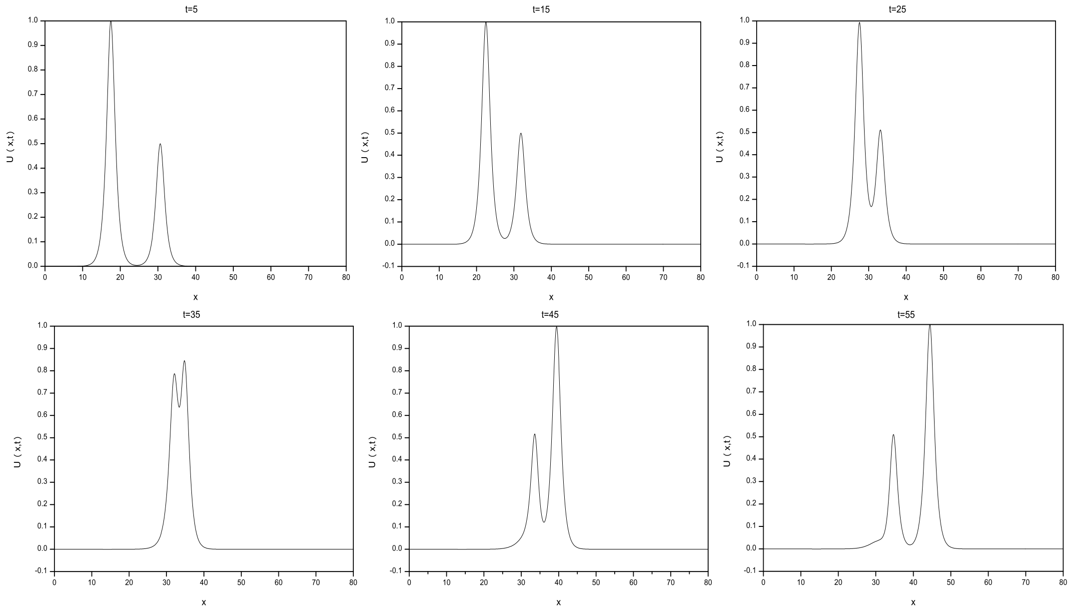

Figure 2: *Interaction of two solitary waves at different times* $h = 0.1, \Delta t = 0.025, A_1 = 1, A_2 = 0.5, (0 \leq x \leq 80)$.

are found to be very close with the analytic values.

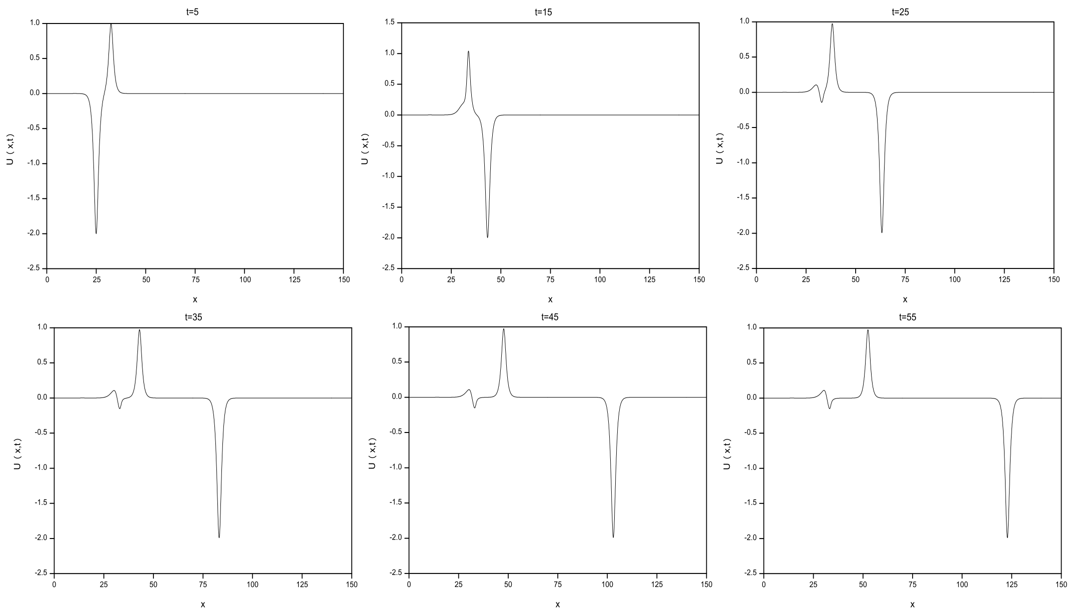

Figure 3: *Interaction of two solitary waves at different times* $h = 0.1, \Delta t = 0.025, A_1 = -2, A_2 = 1 (0 \leq x \leq 150)$.

4 Conclusion

In this paper, a numerical method based on a Petrov-Galerkin method using quadratic weight functions and cubic B-spline finite elements has been presented to find numerical solutions of MEW equation. We tested our scheme through single solitary wave in which the analytic solution is known and extended it to study the interaction of two solitary waves where the analytic solution is unknown during the interaction. The performance and accuracy of the method were shown by calculating the error norms L_2 and L_∞. Also, the obtained results are much better than those found by [10, 11, 12] and in good agreement with [8]. So the obtained results show that a Petrov-Galerkin method involving quadratic weight functions and cubic B-spline finite elements can be used to produce reasonably accurate numerical solutions of the MEW equation. So, this method is a reliable one for getting the numerical solutions of the physically important non-linear problems.

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