Journal for the Education of Gifted Young Scientists, 11(4), 541-556, Dec 2023
e-ISSN: 2149- 360X
jegys.org
dergipark.org.tr/jegys

Young Wise Publishing
youngwisepub.com
gencbilgeyayincilik.com
© 2023

Research Article

Subject, functionality and level of proofs preferred by pre-service elementary mathematics teachers

Bahar Dinçer[1*] and Deniz Kaya[2]

Department of Mathematics and Science Education, Izmir Demokrasi University, Izmir, Turkiye

Article Info

Received: 25 September 2023
Accepted: 15 December 2023
Available online: 15 Dec 2023

Keywords
Mathematics education
Pre-service teacher
Proof
Trigonometry

Abstract

This study examined the subject(s) that elementary mathematics teacher candidates find most suitable for proving in analysis courses, the functional structure of proof they remember most, the level of proof, and the reasons for preferring this proof. In this study, which was conducted with a qualitative research approach, a form consisting of open-ended questions was applied to teacher candidates. In this form, teacher candidates were asked questions about the mathematical proofs they made. With descriptive analysis, the answers of the pre-service teachers who participated in the research were systematically defined, and data were tried to be determined through content analysis. Accordingly, while the pre-service teachers found the most appropriate application of the proof approach to be the subject of trigonometry, it was determined that the proof that remained in their minds the most was also related to the topic of trigonometry. By examining the functional structure of these proofs written by pre-service teachers, it has been seen that they have the function of explanation and systematization. In addition, the reasons for preferring the proof they made were asked of the pre-service teachers, and the answers were gathered on the fact that proof provides the most permanence and causal learning. It was emphasized that theorems that require formula memorization generally become more understandable with the proof method. According to the results of the research, the common opinion of the pre-service teachers is that teaching how to obtain the proof method of formulas in trigonometry instead of memorizing them is beneficial in ensuring both meaningful and permanent learning. In light of the findings of these studies, more sensible suggestions can be made to improve pre-service teachers' knowledge systems and classroom teaching on proof. By determining which topics and theorems pre-service teachers have difficulty in proving in addition to trigonometry, additional learning on these subjects can be recommended.

To cite this article:

Dinçer, B. & Kaya, D. (2023). Subject, functionality and level of proofs preferred by pre-service elementary mathematics teachers. *Journal for the Education of Gifted Young Scientists*, *11*(4), 541-556. DOI: http://dx.doi.org/10.17478/jegys.1365213

Introduction

There are many theorems in the light of operational and conceptual knowledge in mathematics education. These theorems are not just a formula but represent a semantic whole based on different mathematical ideas and propositions. A mathematician uses valid logical inferences rather than empirical and observational results to demonstrate the truth of a mathematical statement and proposition (Hanna & Barbeau, 2002). All of these logical inferences lead us to the

[1] Assistant Professor Dr., Department of Mathematics and Science Education, Izmir Demokrasi University, Izmir, Turkiye, bahar.dincer@idu.edu.tr ORCID: orcid.org/ 0000-0003-4767-7791
[2] Associate Professor Dr., Department of Mathematics and Science Education, Nevşehir Hacı Bektaş Veli University, Nevsehir, Turkiye, denizkaya@nevsehir.edu.tr ORCID: 0000-0002-7804-1772

concept of mathematical proof. Mathematical proof includes revealing relationships, making predictions, relating concepts, verifying statements, and generalizing new information (Schabel, 2005). Proving has an important place in mathematics education. The primary purpose of advanced mathematics courses is to provide students with the ability to prove theirs. Students' proficiency in proving is seen as an assessment of their performance (Weber, 2001). Proof has different educational functions. Mathematical proof helps to understand the meaning of the given theorem and shows both the correctness of the theorem and why it is true (Hanna, 2000). Also, proofs have different benefits and functions. Some of these benefits are helping to create good definitions and practical algorithms contributing to systematizing results and formalizing mathematical knowledge. There is a systematic classification in the literature for the different proof functions. The following is a valuable list of the functions of proof and proving (de Villiers, 1990, p.5):

Verification (concerned with the truth of a statement): One of the proof functions that mathematicians are most familiar with is verification. Accordingly, a statement or proposition is only considered a theorem if its correctness has been demonstrated. Verification is understood as demonstrating the truth of a mathematical proposition. Many mathematicians believe that the proof provides absolute certainty and is the absolute authority in ensuring the claim's validity. The experiment has an important place in ensuring the accuracy of scientific facts, and the proof has an important place in ensuring the accuracy of mathematical facts. However, while proof is not a necessary prerequisite for verification, verification is a prerequisite for proof.

Explanation (providing insight into why it is true): The role of proof and explanation is generally emphasized in the lessons. In most mathematical content, proof should explain why a proposition is true rather than show that it is true. On the other hand, although a claim's validity can be mainly ensured by verification, more is needed as to why the claim is valid. In this context, the explanatory function of the proof provides sufficient confidence about why the claim is valid. However, limiting this role of proof only to course content is incorrect.

Systematization (organizing various results into a deductive system of axioms, significant concepts and theorems): This proof function is mainly encountered in Euclid's book Elements. In this book, many theorems that Greek mathematicians proved are collected and arranged as theorems, definitions, axioms and postulates. There are also enough definitions, axioms, and postulates to develop Euclidean geometry. In addition, the first essential results of number theory, such as divisibility rules, prime numbers and factorization, are also included. By systematizing mathematics, mathematicians can eliminate the unnecessary redundancy of definitions and axioms and have the opportunity to capture all the necessary concepts and ideas with a small set of them.

Discovery (the discovery or invention of new results): This function of proof is rare. Historically, theorems of some areas of mathematics, such as non-Euclidean geometries, have been arrived at through abstract deductive reasoning. For example, in Euclid's fifth postulate of parallelism, non-Euclidean geometries emerged through theorems describing the geometry of shapes in curved surfaces rather than planes.

Communication (the transmission of mathematical knowledge): The communication function of proof refers to the reading and writing of proofs by people. According to this function, proofs are seen as a means of communicating mathematical results among people. Moreover, proofs can offer a new approach or technique, providing opportunities for other mathematicians to complete and develop a different theorem(s) of their own.

Intellectual challenge (self-realization derived from constructing proof): For mathematicians, proof is an intellectual challenge. In this sense, proof serves the function of self-actualization and realization.

The functions mentioned above of mathematical proofs may differ depending on the person who makes and reads the proof. In this context, the purpose and proficiency of a mathematical proof differ. Although mathematical proof is one of the essential concepts in mathematics that distinguishes mathematics from other sciences, it is one of the concepts that students have difficulty understanding (Arsac, 2007). Proving is at every level of education. Although the concept of mathematical proof is central in university-level mathematics courses, research indicates that some students at this level need help to understand what proof includes and how it is developed (Jones, 2000).

Literature Review Related to the Research Topic

When the literature is examined, many studies are on levelling the arguments created by students (Balacheff, 1998; Knuth et al., 2009; Miyazaki, 2000). Researchers defined different levels for the proofs made by individuals and classified the proofs made according to these levels. For example, Balacheff (1998) introduced three levels of proof; pragmatic proofs, intellectual proofs and demonstrations. In pragmatic proof, which is the lowest level of proof, representations are made with representations of mathematical objects. In intellectual proof, the statements in the question and the relationships between these statements are formulated. At the demonstration level, information explained by a theory or accepted by a community is used. Miyazaki (2000) divides proof into four groups proof A, proof B, proof C, and proof D. Proof A represents the most advanced proofing skill. Proof C represents the lowest category in the proof-making process. Proof B and Proof D are intermediate between Proof A and Proof C. The use of language, inductive reasoning and method preferences effectively made this classification. Knuth et al. (2009) divided individuals' proofs into four levels. Level 0 individuals need to be made aware of using mathematical proofs to demonstrate the correctness of the situation. For example, a student accepts that situation because his teacher says it is true. The level 1 individual is aware of the need for proof, but in this process, they only verify the situation by using special situations. Level 2 individuals, on the other hand, either need to be able to reach the generalization by using a wrong method in the generalization process or completing the proof correctly. Individuals at the highest level, level 3, generalize and reveal the truth of a statement with appropriate methods and arguments. Along with all these studies, when studies examining pre-service teachers' proving levels are examined, it is stated that pre-service teachers' level of proving is low (Jones, 2000; Weber, 2005). In the literature, it is possible to find studies not only about the level of proof but also about all proof processes in general. Proving has been seen as a process in which students cannot be successful or a process in which they believe they will not succeed at every stage of education (Morris, 2002; Raman, 2003). Recent studies documenting teachers' fragile understanding of proof and how it progresses show that improving the role of proof in school mathematics should significantly support teacher learning (Lesseig, 2016). In addition, researchers state that one of the ways to reach classroom practices and student learning underlined by educational reforms is to change the course materials used during the course (Cai & Howson, 2012). According to this result, we again come across teacher competencies. In that case, the emergence and development of the ability to prove to lie in that teachers create opportunities for students to improve these skills (Bieda et al., 2014). Along with this requirement, it is seen that proof studies reveal that students have difficulties in proving (Stylianides & Stylianides, 2023; Urhan & Bülbül, 2016; Weber, 2001). In one of these studies, the current difficulties were explained in detail. It was revealed that when the pre-service teachers were asked to "prove or show that it is true", they did not question, did not have knowledge about alternative proof methods, and insisted on using the proof methods they were familiar with (Demircioğlu, 2023, p. 331). In a different study conducted with teachers and students on eliminating these difficulties, it was stated that visual proof provides a better learning experience (Polat & Akgün, 2023). However, to date, little research details what mathematical knowledge can support teaching proof or how professional development can accommodate such learning. This article examines the different aspects of proof making and wishes to present a holistic result that combines these details.

Purpose and Importance of the Research

In the last part of this study, it was mentioned that proof has an important place in mathematics, together with the studies on the purpose of proof, levels of proof and inadequacies. Since proof is a multidimensional process, studies investigate different parameters related to proof. Along with this high degree of importance of proof, different studies examine the functionality of proof and skills of proving. A literature review conducted for the analysis course examined the characteristics of proofs and correct proof ratings (Doruk & Kaplan, 2017; Kotelawala, 2007; Saeed, 1996). Differently from the literature, in this study, the subject that the students found the proof to be most appropriate, their level of proof, the functionality of the proof that remained in their minds the most, and the reason for choosing this proof were examined. Because one of the purposes of proof is to provide meaningful and permanent learning, for this reason, the students were asked only the proof that remained in their minds without any guidance or proof of a theorem. Thus, the functional structure of the proof that has the most permanence effect was sought to be examined. Although there are different mathematics courses at the university level, one of the most important courses in the curriculum is analysis (Hartter, 1995). Therefore, the research is designed for the proofs in the analysis course because there are many mathematical proofs and fundamental theorems in the analysis course. In connection with this, it was observed by the researchers that the students in the analysis courses at the university needed help in the process of proving.

For this reason, researchers left the choice of subject or theorem to students to prove. Thus, they wanted to examine the proof preferences of the students without any guidance. In particular, they wanted to observe whether their students needed help proving the subject/theorems they chose according to their preferences. Moreover, the students were asked why they preferred the proof they chose. Thus, it is aimed to establish a connection between the most memorable proof, its functionality, its subject and the reason for preference. The difference between this study from other studies in the literature is that there is no question or theorem orientation in the proof processes for students. In addition, when the literature is examined, it is possible to find many studies on levelling the arguments created by students (Balacheff, 1998; Bell, 1976; Harel & Sowder, 1998; Healy & Hoyles, 2000; Quinn, 2009). However, in this study, along with the levelling of proof, the purpose of proof has also been examined in integrity. In this context, it will bring a different perspective to the literature.

In this study, which was carried out in light of all this information, the subject that the pre-service teachers found most suitable for proof in analysis courses, the functional structure of the proof that remained in their minds the most, and the reasons for preferring this proof and the level of proof were examined. In addition, whether there is a relationship between the subject that pre-service teachers prefer to prove the most, the proof that remains in their minds the most, and the reasons for choosing proof has been investigated. Regarding the analysis course, the tendency of the pre-service teachers to prove, the subject they prefer to prove, and the choice of proofs with the highest permanence level. It is thought that related studies will contribute to field educators and literature on meaningful and permanent learning, which is the primary purpose of proof. The teaching to be done by considering the pre-service teachers' preferences and thoughts on the proof will be more beneficial than the current teaching.

Method

Research Model

Within the scope of this research, while examining the opinions of the pre-service teachers about the subject they prefer to prove, the functionality of the proof they remember the most, and the reasons for preferring proof, case study, one of the qualitative research approaches, was used. The essential feature of the case study is the in-depth investigation of one or more cases. In other words, the factors related to a situation (environment, individuals, events, processes, *etc.*) are investigated with a holistic approach and focused on how they affect the relevant situation and how they are affected by the relevant situation (Yıldırım & Şimşek, 2008). This study aims to reveal the pre-service teachers' preferences for proof in analysis courses, the proof they keep in mind the most and its functional structure, their level of proof, and their views on proof relationally and holistically. In this case, this study focused on more than one situation. According to Yin (2003), when the case study includes more than one analysis unit, it focuses on the sub-unit or sub-units within a situation. This situation emerges in nested case studies. Yin (2003) defines a case study as a research method that is used when i) the research focuses on "how" and "why" questions, ii) the researcher has little or no control over events, and iii) the event or phenomenon is studied within the framework of its natural life. In this study, the participant's level of proving, the function of their proofs, their subject preferences in the proof and the reasons for these preferences were examined holistically without giving any direction to the participants. Therefore, this study is defined as a nested case study. The situations in this study are the pre-service teachers' preferences for proof in analysis courses, the most memorable proof, its functional structure, the level of proof and their views on proof and the relationship between them.

In this study, all rules stated to be followed within the scope of "Higher Education Institutions Scientific Research and Publication Ethics Directive" were followed. None of the actions stated under the title "Actions Against Scientific

Research and Publication Ethics", which is the second part of the directive, were not taken. The study was approved by the Izmir Demokrasi University Human Subjects Research Ethics Committee (Date: 8 April 2022, Number: 22/04-03).

Participants

The study group of the research consisted of four pre-service teachers studying in the department of elementary mathematics education in the 2021-2022 academic year. A purposeful sampling method was followed to conduct interviews. Purposeful sampling methods emerged within the qualitative research process. Purposeful sampling allows for an in-depth study of situations that are thought to have rich information (Yıldırım & Şimşek, 2008). In this sampling, criteria that are considered essential for selection are determined. It is thought that the sample selected according to these criteria can represent the research population with all its qualities (Tavşancıl & Aslan, 2001). It was thought that interviewing teacher candidates whose letter grades in Analysis 1-2 courses were AA-BA was important in reflecting their views on proof. Because the studies in the literature stated that proving is a complex process and is closely related to the academic success of the students (Arslan, 2007; Çalışkan, 2012), therefore, it was thought that it would be more accurate to select students with high academic achievement as the study group. Students with high academic success in the analysis course were selected by examining the grade list of the previous semesters. As a result, it was thought that interviewing the pre-service teachers whose letter grades in Analysis 1-2 were AA-BA was important in reflecting their views on proof. These elections were made voluntarily.

Data Collection and Analysis

In the study, in terms of holistic analysis, the subject that the pre-service teachers found appropriate to be explained with proof within the scope of analysis 1-2 course and the reason for their choice of proof and preference were asked with the help of a form consisting of open-ended questions. It is foreseen that this form will be completed in approximately 30 minutes, and therefore it was decided to give this time to the participants in this study. In this form, three questions were asked of the pre-service teachers.

- Which subject is most suitable for proof within the scope of analysis courses?
- Which proof remains in your mind the most within the scope of analysis courses?
- Complete the sentence "I chose this proof because...". What is your reason for choosing proof within the scope of analysis courses?

These statements were determined by the researcher and a mathematics education expert, and it was agreed that the questions were suitable for the content and purpose of the research, and content validity was ensured. The mathematics education expert was an educator with a master's degree in mathematics education. Coding reliability was calculated using the percent of agreement index. Percent of agreement is an index found by calculating the ratio of the cases in which the same coding is made to all existing cases. In this way, the coding reliability was found to be 0.91 using the percentage of agreement. The percentage of agreement is expected to be higher than 70% (Tavşancıl & Aslan, 2001). The result obtained is an appropriate percentage for this study. The data obtained at the end of the research process were subjected to descriptive and content analysis. With descriptive analysis, the answers of the pre-service teachers who participated in the research were systematically defined, and data were tried to be defined through content analysis; The data, which were found to be similar and related to each other, were brought together and interpreted within the framework of specific concepts and themes. The functional structure of the proof, which the pre-service teachers most remember, was examined categorically according to the purposes of the proof stated in Hanna's (2000) study. Because it is essential for the result of the research to determine for what purpose the pre-service teachers did the proof. Thus, it is planned to establish a holistic relationship between the subject of the proof and the purpose of the proof. The data on the functional structure of the proof are given in Table 1.

Table 1. Functions of proof

Function Themes
Verification (concerned with the truth of a statement)
Explanation (providing insight into why it is true)
Systematisation (the organization of various results into a deductive system of axioms, major concepts and theorems)
Discovery (the discovery or invention of new results)
Communication (the transmission of mathematical knowledge)
Intellectual challenge (the self-realization/fulfillment derived from constructing a proof)

A variable that the research deals with are the level of proof of pre-service teachers. The data obtained from the research were analyzed descriptively with a rubric prepared according to the proof levels in the study of Knuth et al. (2009). Detailed information about these levels is given in Table 2.

Table 2. Categories of levels of proof

Levels	Indicators
Level 0	Writing mathematical statements that are not intended to generalize or accept the accuracy of the given statement without any explanation
Level 1	Verifying the proof by using extreme (special) cases in the mathematical proof process
Level 2	Writing generalizing but mathematically inappropriate statements and incomplete proofs
Level 3	Evidence acceptable to authorities

The data obtained from the teacher responses were expressed as frequency/percentage values under categories. Since this study is qualitative since the sample was not chosen to represent the population, statistical generalization was not made; instead, the findings were evaluated to explore the examined subject in depth. The researchers stated that the names and information of the pre-service teachers would not be shared with anyone. As a result of the research, it was stated that the codes would be used instead of the participants' names in the article (P1, P2, P3, P4). In addition, for the research's validity, the pre-service teachers' views were included with one-to-one quotations. Overall, through an analysis of the data collected, this study aims to answer the following research questions:

- Which subject do pre-service mathematics teachers find most appropriate to prove?
- What subject of proof remains in the minds of pre-service mathematics teachers the most?
- What function of proof remains in the minds of pre-service mathematics teachers the most?
- What level of proof remains most in the minds of pre-service mathematics teachers?
- What is the reason why pre-service mathematics teachers preferring to prove?

Results

This study aims to examine the subject that pre-service teachers find most appropriate to prove, the function of proof that remains in their minds the most, and the reasons for preferring proof. In this context, the findings obtained in accordance with the sub-problems determined within the scope of the study are presented respectively. The answers given by the participants to the questions were first analyzed in the context of the topics. Table 3 contains data on the subjects that pre-service teachers find most appropriate to prove.

Table 3. Topics where proof is preferred

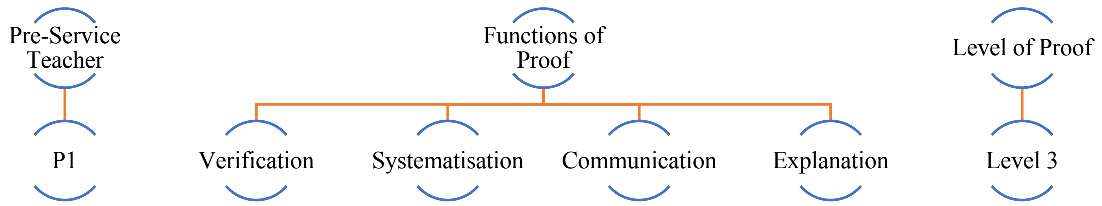
Topics preferred for proof	f (%)
Trigonometry	4 (100)
Derivative	3 (75)
Integral	2 (50)

According to the data in Table 3, pre-service teachers prefer proving trigonometry within the scope of analysis one and two courses. Derivative and integral subjects, which are among the subjects of the analysis course, are also among the subjects that are suitable for proof. Three pre-service teachers answered derivative, and one student answered integral. In Table 4, it is given which subject the proof belongs to the most in the minds of the pre-service teachers.

Table 4. The subject of the proof that remains in the minds of the pre-service teachers the most

Topics preferred for proof	**f (%)**
Trigonometry	4 (100)
Other topics	0 (0)

According to the data in Table 4, when the proof that the pre-service teachers most remembered was examined, it was determined that all answers were about trigonometry. Below, the images of the most common proofs in pre-service teachers' minds were presented, and the functions and levels of these proofs were examined. These functions were determined according to the categories de Villiers (1990) stated in his study. In the content of the table, information is also given about the level of proof made by pre-service teachers. Proofs of pre-service teachers were categorized according to the levels determined by Knuth et al. (2009). The most memorable proof of the pre-service teacher (P1) was shown in the figure below, then the functions and level of this proof were presented in Category 1.



Category 1. Function and level of proof of participant P1

The first participant preferred a general explanation regarding the accuracy of the statement made. Immediately afterwards, the participant followed the step of organizing the various results into a deductive system consisting of axioms, basic concepts, and theorems. In the next step, the participant attempted to transfer mathematical knowledge. While doing these, the participant tried to prove why it was correct by using visuals. The participant's solution process is presented below.

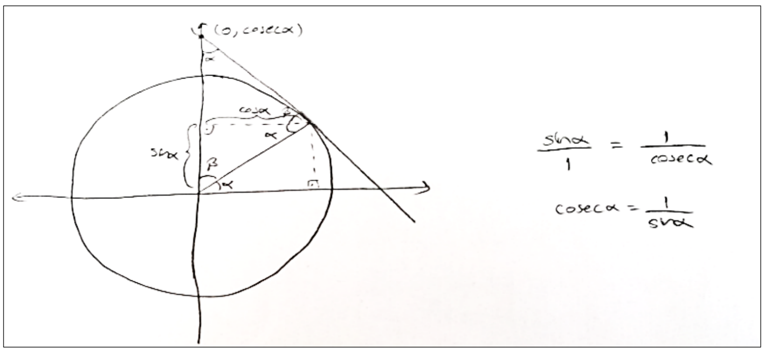
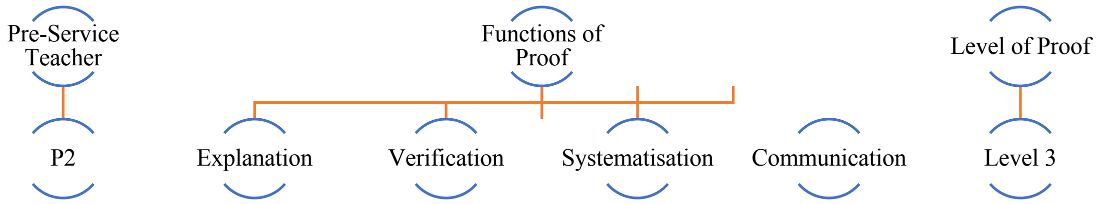


Figure 1. The proof that the pre-service teacher-P1 remember the most

The proof that pre-service teacher P1 remembered the most was the subject of trigonometry. P1 proved the expression of the cosecant function as a 1/sine function on the unit circle. While making this proof, he also benefited from the similarity in triangles. This proof included verification, systematization, communication, and explanation. In addition, when examined in terms of proof level, it was determined that it was the highest level (level 3) of proof. Because in this proof, a proof accepted by the mathematical authorities of a trigonometric function was made using mathematical connections. In the continuation of this study, pre-service teachers were asked to complete the sentence, "I preferred this proof because ..." regarding the reasons for choosing the proof that remained in their minds the most. Moreover, the reasons why pre-service teacher P1 prefer proof are given below.

"I preferred this proof because it provides a better understanding of the basics rather than memorization. I've always memorized the cosec function as the multiplicative inverse of the sine function until now. When I said the cosec function, I always thought of the 1/sine function, but I didn't know why. I learned the reason for this in the lesson on the unit circle and a more meaningful learning took place for me. Thus, I learned how to obtain a formula that I memorized, and this proof remained in my mind. The important thing is not to memorize, but to learn meaningfully. When I'm a teacher, telling my students to "memorize the formulas" is an easy, but not effective, method. Instead, explaining which formula came from where and how provides students with more permanent learning. At the same time, they can have a positive attitude towards the mathematics lesson."

The most memorable proof of the pre-service teacher (P2) was shown in the figure below, then the functions and level of this proof were presented in Category 2.



Category 2. Function and level of proof of participant P2

The second participant also preferred to explain where a generally accepted formula came from in his proof. The participant took care to act within a process while using the formula. In this process, the participant used a visual approach-based theorem proof and reached the conclusion through a gradual process. At the same time, the participant also benefited from his previous mathematical knowledge. The participant tried to prioritize mathematical knowledge, but this remained limited. While using mathematical knowledge, the participant preferred the visualization method and tried to prove it. The participant tried to support what the formula expressed with a visual. The participant's solution process is listed below.

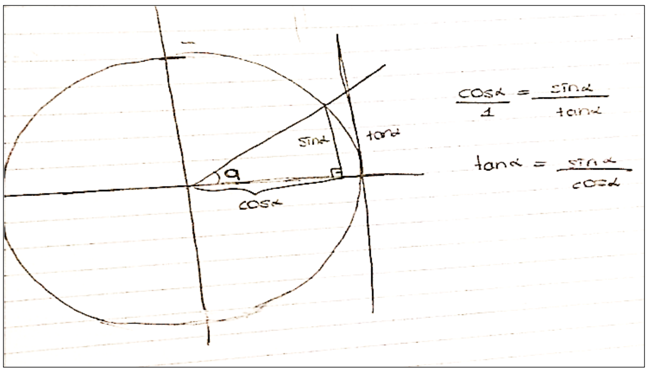
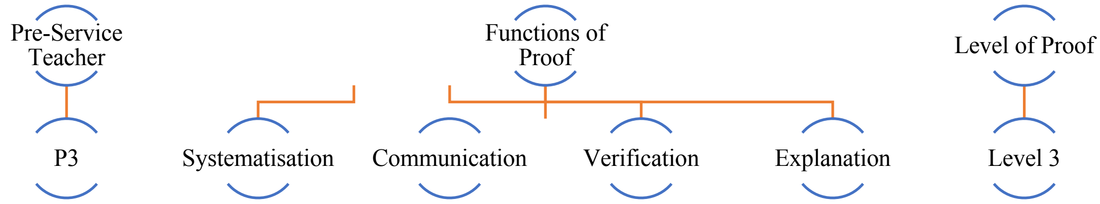


Figure 2. The proof that the pre-service teacher-P2 remember the most

The proof that pre-service teacher P2 remembered the most was again about trigonometry. P2 proved the proof of writing the tangent function as the ratio of the sine and cosine functions on the unit circle. While doing this proof, s/he obtained a ratio by using the similarity in triangles. This proof included verification, systematization, communication, and explanation. In addition, when it is examined in terms of proof level, it has been determined that it is proof at the highest level. In this proof, it was stated how a well-known trigonometric ratio formula is obtained by using mathematical connections. For this reason, it has been determined that this proof is level 3. The pre-service teacher (P2) was also asked why s/he preferred this proof, and s/he gave the following answer.

"I preferred this proof because it allows thinking about the basis of that knowledge, not memorizing it. One of the most frequently used formulas in trigonometry is the tangent function. The tangent function is given not only as a slope, but also as a sine/cosine function. And so far, all my math teachers have said that the tangent function is equal to sine/cosine, but they didn't say why. I memorized this ratio as my teachers told me. But later I saw that this formula had a very simple proof on the unit circle, and so it stayed in my mind. Now it made more sense for me when I said sine/cosine to the tangent function. Now, when the tangent function is called, it makes more sense for me mathematically. Otherwise, I was saying sine/cosine, but saying that doesn't mean I understand the concept of tangent. On the contrary, it only shows that I learned superficially. The purpose of learning is permanent learning. Now I feel more confident in trigonometry subjects."

The most memorable proof of the pre-service teacher (P3) was shown in the figure below, then the functions and level of this proof were presented in Category 3.



Category 3. Function and level of proof of participant P3

The third participant explained a trigonometric formula using another trigonometric ratio and proved where this explanation came from. While doing this, care was taken to act within a certain system. The participant used visual and algebraic representations here and also explained how he identified similar triangles as a prerequisite before doing the proof. While doing this, the participant tried to explain his proof by using visuals. The participant's solution process is listed below.

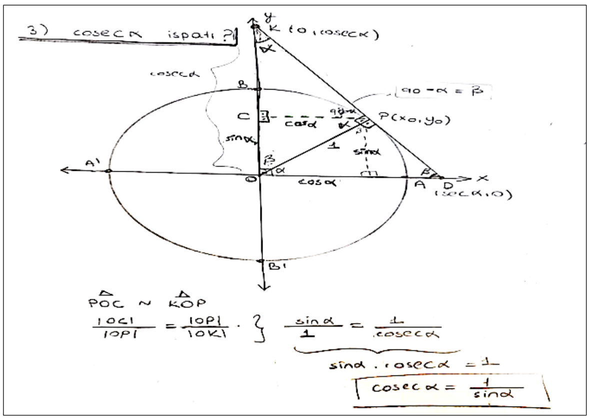
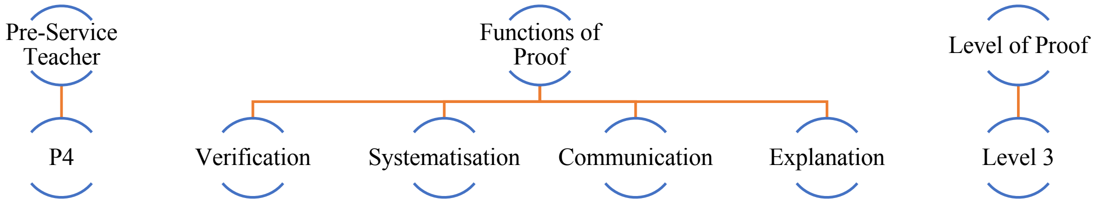


Figure 3. The proof that the pre-service teacher-P3 remember the most

As with the other participants, the proof that pre-service teacher P3 had the most in mind belonged to the subject of trigonometry. P3 proved the proportional formula of the cosecant function. Pre-service teacher P3 preferred to prove the cosecant function, like pre-service teacher P1. As can be seen from the answers, it was seen that the candidate numbered P3 made a more ample proof and wrote the data about the theorem more comprehensively. This proof includes verification, systematization, communication, and explanation. In addition, when examined in terms of proof level, it was determined that it was proof at the highest level. This proof clearly stated mathematical connections and how a commonly used ratio was obtained were expressed. For this reason, it has been determined that this proof is level 3. The pre-service teacher (P3) was also asked why s/he preferred this proof and s/he replied as follows.

"I preferred this proof because it provides meaning rather than memorization The Cosecant function is always expressed as a 1/sine function. I memorized it this way, without questioning why. Then I saw that this proof could be made by making use of the similarity theorem. Maybe I couldn't have done this proof by myself, but after seeing it that way in class, it stuck in my mind. Learning trigonometric formulas with their proofs provided me with a much more permanent learning process. I think I will be more successful if I learn all math topics and formulas like this. For example, normally it is difficult for me to prove, but even proving is easy when I learn meaningful and step-by-step. At the very least, proof has been a tool for meaningful learning for me. Proving for trigonometry is meaningful, I hope I can have such a meaningful learning process for other courses and subjects."

The most memorable proof of the pre-service teacher (P4) was shown in the figure below, then the functions and level of this proof were presented in Category 4.



Category 4. Function and level of proof of participant P4

The fourth participant, unlike the other participants, did not prefer to use the unit circle in his proof. The participant did not prove a trigonometric formula, but a derivative of a trigonometric formula. While performing this proof, the participant benefited from his prior learning and different applications on derivatives (chain rule). At the same time, the participant also used trigonometric relations in the right triangle and expressed them visually. While making his statements, the participant tried to act in a certain order and tried to use all the components together to present his proof. The participant's solution process is listed below.

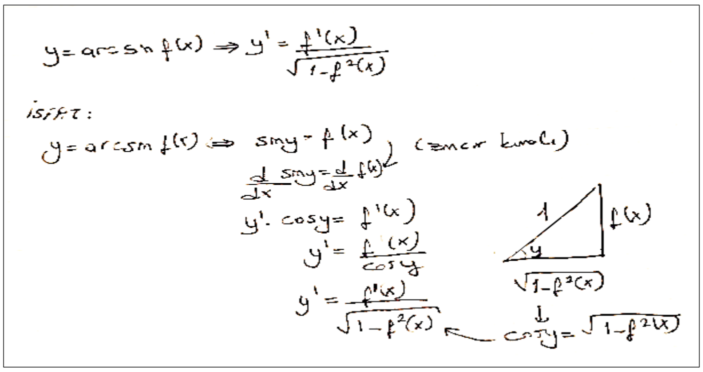


Figure 4. The proof that the pre-service teacher- P4 remember the most

The proof that the pre-service teacher P4 remembered the most was the subject of trigonometry, as was the case with the other participants. Here, the derivative formula of an inverse trigonometric function was proved instead of an essential trigonometric relation. Pre-service teacher P3 also benefited from the Pythagorean theorem while proving. This proof included verification, systematization, and communication functions. The function "Exploration of the meaning of a definition or the consequences of an assumption "is not included in this proof. Because here, a proof of a formula was made instead of a definition or a well-known ratio. Unlike a general trigonometric ratio in the proofs of pre-service teachers P1, P2, and P3, this proof included the derivative relation. It has been determined that this level of proof is the highest as in other proofs. In this proof, it was expressed how a commonly used formula was obtained by specifying mathematical equations. For this reason, it was coded as level 3. The pre-service teacher P4 was asked why s/he preferred this proof, and the following answer was obtained from his/her.

"I preferred this proof because it helps me understand a formula that is hard to memorize. Actually, the formulas of inverse trigonometric functions have always been more difficult for me. But now I understood the derivative formula of the arcsine function more clearly and this proof stuck in my mind. Thus, I learned a formula not as a pattern, but by knowing where it came from. And as such, inverse trigonometric functions and their derivative calculations have become more meaningful to me. What I expect from my teacher is to tell me where a formula comes from in the easiest way possible. The formulas that I just memorized without understanding the reason caused me to have a negative attitude towards mathematics and sometimes I was afraid of mathematics lessons. Or I just memorized formulas to pass the exam. Now I see that it wasn't that hard to actually learn meaningful. I don't know if I can learn every subject like this, but it made me very happy to prove something that I thought difficult..."

As can be seen in Figures 1, 2, 3 and 4, the proofs that pre-service teachers remember the most were about trigonometry. All participants provided proof of trigonometry. In the proofs of the pre-service teachers, in addition to the verification, explanation and communication functions of the proof, "systematization and exploration of the meaning of a definition" functions were also identified. In addition to the verification, explanation, communication and systematization steps, the exploration of a defined function's meaning is frequently used in proofs for trigonometric concepts (for example, cosecant, tangent), was observed. It was possible to see this function in the proofs of pre-service teachers P1, P2, and P3. In addition, pre-service teachers P1, P2, and P3 used the unit circle in their proofs. Pre-service teachers P1 and P3 preferred to prove the expression 1/sine(x), which is the rule of the cosecant function. The pre-service teacher P2, on the other hand, explained how to find the sine/cosine ratio, which is an expression of the tangent function on the unit circle. They also stated these by citing the similarity theorems on the unit circle. Pre-service teacher P4 preferred to prove the derivative of a function. The P4 pre-service teacher also preferred the inverse of the sine function, a trigonometric function. P4 proved where the derivative formula of the arcsine function came from; s/he also used different theorems in the triangle in this proof. Moreover, thus s/he proved where the rule of the arcsine function comes from.

When the proving levels of the pre-service teachers were examined, the proving levels of the pre-service teachers P1, P2, P3 and P4 were determined as the 3rd level. Because all of the proofs in this study were highly cognitive proofs accepted by mathematical authorities, in these proofs, the cognitive basis of each proof was presented by making use of similarity in triangles, unit circles or different theorems in triangles. In the interviews with the pre-service teachers, they expressed their proof verbally with their support and gave the necessary arguments. In the continuation of this study, pre-service teachers were asked to complete the sentence, "I preferred this proof because ..." regarding the reasons for choosing the proof that remained in their minds the most. A common theme was determined in the answers of the pre-service teachers P1, P2, P3 and P4. This theme was that they wanted to avoid memorizing while learning mathematics. When the answers of all pre-service teachers were examined, it was seen that they generally used the word memorization and preferred to prove instead of memorizing. All the proofs in this work were on a trigonometric ratio given as a formula or on explaining a formula. However, pre-service teachers proved where these formulas came from with scientific grounds. They also stated that the reason they preferred proof was that they did not want to memorize it. In addition, it was seen that the reasons given by the pre-service teachers for preferring proof were in accordance with the function of proof. In this respect, harmony was observed between the functionality of the pre-service teachers' proofs and the reason for preferring these proofs.

Conclusion and Discussion

This study examined the subject that the pre-service teachers find most suitable to prove, the function of the proof they remember the most, and the reason for choosing this proof and their proof level. In the findings section, when the subjects that the pre-service teachers found most suitable for proving were examined, trigonometry, derivative and integral subjects were preferred, respectively, according to the number of answers the participants gave. At this point, the pre-service teachers found it appropriate to prove the concepts of trigonometry, derivative and integral, which are the main subjects of the analysis course. All of the pre-service teachers in this study answered trigonometry in the first place for the subject they found most suitable for proving. All pre-service teachers who participated in the research agreed on trigonometry in connection with the fact that the subject of trigonometry contains many formulas.

Within the scope of the current study, a similarity was also found between the subject of proof that the pre-service teachers remember the most and the topics that they find the most appropriate to apply proof. Trigonometry was again the subject of proof that the pre-service teachers remembered the most and the subject they were most suitable for proving. The fact that the most memorable proof and the subject suitable to be proved are similar can be expressed holistically with both permanence and causality principles. In addition, the most memorable proof about trigonometry can be explained by the fact that pre-service teachers were familiar with trigonometric formulas at the elementary school level and frequently encountered proofs on trigonometry in general. In that case, for pre-service teachers to prove about a subject and provide permanent learning with proof, preliminary knowledge about the subject should be sufficient. In different studies, pre-service teachers were asked to prove certain theorems on certain subjects, and as a result, it was concluded that pre-service teachers were insufficient in proving (Almeida, 2000; Jones, 2000; Moralı et al., 2006; Stylianides & Stylianides, 2023; Urhan & Bülbül, 2016; Weber, 2001). However, in this study, the subject/theorem to be proved was left to their preferences, without any subject or theorem orientation to the pre-service teachers.

Moreover, it was seen that all pre-service teachers participating in the research made proof within the scope of the analysis course. The pre-service teachers' proving proficiency in this study is different from other studies because the pre-service teachers have sufficient prior knowledge about the subject they prefer to prove. Mathematics is a reason-based science, so proof-making is very important in mathematics. However, it is seen that many teachers need more content knowledge on this issue, which constitutes the logical foundations of proof (Healy & Hoyles, 2000; Knuth, 2002; Martin & Harel, 1989). In that case, the prerequisite for students and pre-service teachers to succeed in the proof is sufficient prior knowledge of mathematical concepts and subjects. However, this situation should only be considered a prerequisite in proof with generalization because general mathematics achievement differs from having sufficient content knowledge about proof (Jones, 1997). According to the research results, although the prerequisite for making proof is mathematical prior knowledge and success, proving and mathematical success are separate parameters. Mathematical success and prior knowledge can only be seen as prerequisites in proving.

As a result of the research, it was another remarkable finding that the proof that remained in the minds of the pre-service teachers mainly consisted of visual proofs. In that case, it would be beneficial to increase the visual representations in the teaching content, including the visual proofs, to ensure permanent and meaningful learning. In different studies in the literature, the effect of the stimuli factors in the learning environment on the learner's permanent learning has been revealed (Angelides & Agius, 2002; Botana & Valcarce, 2002; Strijbos et al., 2003). A similar result in this study regarding the importance of visual proof was obtained in another study, and it was stated that visual proof provided a better learning experience (Polat & Akgün, 2023). In other words, the importance of visual proof has been reinforced with this study. In the proofs of the pre-service teachers, in addition to "the verification, explanation and communication functions of the proof, systematization and exploration of the meaning of a definition", functions were also identified. Schoenfeld (1994) states that in traditional teaching environments, the explanatory power of proof is not utilized in favor of students. Therefore, students need help to capture the depth of thought inherent in proof. Another essential role of proof in mathematics education is that it enables sharing of information and ideas; that is, it functions as a communication tool. As mentioned in the previous section, many mathematics educators define proof as a social activity that allows sharing of ideas between individuals (Hanna, 1990) and as a product of the mathematics learning-teaching process (Wheeler, 1990). In the proof process, it is clear that a cognitive-based but predominantly social communication and interaction process takes place.

For this reason, mathematics educators define proof as a social activity and a product of the learning-teaching process and emphasize its contributions to individuals in this respect. As a result of this research, it was seen that the generally accepted formulas on trigonometry and derivatives were expressed clearly with proof. Indirectly, expressions, ratios and formulas, widely used in mathematical language, are explained by making them more understandable with proof. In this respect, the proofs used by pre-service teachers contribute to them in terms of mathematical language. As another parameter in the research, the pre-service teachers' proof level was examined. In many studies in the literature, it has been determined that students are inadequate in proving (Jones, 2000; Morris, 2002; Raman, 2003; Stylianides & Stylianides, 2023; Urhan & Bülbül, 2016; Weber, 2001). Based on these results, the proof levels of pre-service teachers within the scope of this study were at a high level, unlike the studies in the literature. When the proofs of the pre-service teachers in the study were examined according to the four different levels suggested by Knuth et al. (2009), it was determined that all the proofs were logical and holistic depending on the cognitive bases, and it was seen that all the proofs were level 3, which is the highest level. Based on this, it can be said that contrary to the common belief that pre-service teachers' achievement levels in proving are low, their proof levels will be high if there is sufficient prior knowledge. The studies conducted by Martin and Harel (1989) and Stylianides et al. (2007) with pre-service teachers revealed that pre-service teachers verified propositions/theorems by using exceptional cases instead of proofs, and they could not use mathematical language correctly. One of the main reasons for this situation may be the teaching methods used in the lessons. The traditional method follows the order of definition, theorem and proof. However, studies have revealed that peer interaction and in-group discussions increase students' proving skills (Sarı-Uzun & Bülbül, 2013; Weber et al., 2008). The researchers who carried out this study also emphasized the proof of formulas and rules commonly used within the scope of analysis 1 and 2 courses. In addition, they showed trigonometric formulas on the unit circle and taught derivative formulas based on cognitive bases, away from memorization. The fact that the pre-service teachers participating in this research have a high level of proof can be explained in connection with this situation. It can be said that teaching a proof-based and rote-free course in a course increases students' ability to prove themselves. From this point of view, it can be said that preparing course content based on proof and cognitive bases will provide students with a meaningful learning process instead of memorization.

In the continuation of the study, one more question was asked to the participants. The pre-service teachers completed the sentence "I preferred this proof because ..." about the proof that remained in their minds the most, and it was seen that the reason they stated was suitable for the function of the proof they made. All pre-service teachers wrote that it provides meaningful learning instead of memorization as the reason for preferring the proof they have done. The fact that all the proofs preferred by the pre-service teachers contain a formula also confirms a meaningful learning desire instead of memorization. All pre-service teachers preferred to make proof about trigonometry. However, they said they were prejudiced about trigonometry before and just memorized the formulas without making sense of them. Pre-service teachers P1 and P3 proved the cosecant function; they benefited from the similarity theorem in triangles even though they used different visuals. Moreover, they stated they needed to learn why the cosecant function was previously expressed as 1/sine. They also developed a positive attitude towards trigonometry subjects when they learned this proof in the analysis class. Pre-service teacher P2 also proved why the tangent function is equal to sine/cosine. Moreover, he said that he could now prove the tangent as a meaningful formula, not just a memorized ratio, as the reason for choosing this proof. Even though the pre-service teacher P4 chose trigonometry again, s/he proved the derivative of an inverse trigonometric function instead of an essential trigonometric relation. Like the other participants, pre-service teacher P4 emphasized the importance of basic formulas meaningfully rather than memorizing them. S/he stated that this meaning-making process is related to the way of learning in the lesson. Therefore, when the answers of all pre-service teachers are examined, a vital result emerges. Students having an effective learning process and a high level of proof skill depend on how and how and how the subjects are taught. In other words, teachers' teaching methods in the lesson can also affect the quality of their students' learning products. A similar result was obtained in a different study, and the importance of teacher training was emphasized. A similar result was obtained in a different study, and it was stated that improving the role of proof in school mathematics would require teachers' learning (Lesseig, 2016). A similar result was obtained in a different study, and it was stated that improving the role of proof in school mathematics would require teachers' learning. As stated in the literature, the development of student abilities lies in teachers creating opportunities for students to develop these abilities. (Bieda, et al., 2014; Cai & Howson, 2012) A different study in the literature revealed that when pre-service teachers were asked to "prove or show that it is true, " they did not question or have knowledge about proof methods (Demircioğlu, 2023, p. 330). In this study, pre-service teachers proved the basic formulas and stated their reasons for preferring this proof by the function of the proof. In that case, the ability of students to prove can be directly related to how the basic formulas and contents presented in the course are taught.

The findings of this study were obtained from the analysis of 1 and 2-course topics. Conducting similar studies on other subjects within the scope of mathematics will contribute to expanding the results of this research. Another suggestion obtained according to the results of this study is the necessity of emphasizing the proof processes in the course contents to increase the pre-service teachers' level of proof. The researchers in this study presented intense content for the proof of the formulas and theorems in the course content. This situation has also improved their students' success towards proof. In this respect, it is essential to emphasize the processes of making proofs in the course contents and to explain the formulas and theorems with scientific relations. In particular, educators who teach trigonometry subjects can achieve more permanent learning by using the properties of the unit circle and similarity theorems. In this context, in order to be able to do advanced mathematics, learning environments and proof activities that enable pre-service teachers to see their deficiencies and mistakes should be included.

Recommendations

Recommendations for Researchers

The findings of this study were obtained from the analysis of 1 and 2-course topics. Conducting similar studies on other subjects within the scope of mathematics will contribute to expanding the results of this research. For researchers who want to research proof-making skills, it can be recommended to conduct experimental studies because teaching processes are critical in students' proof-making skills. Examinations on the types of proof can be added to the subject of research in this field, and it can be examined which type of proof is preferred by students. Determining which subjects and theorems students have difficulty proving can be recommended for additional learning on these subjects. In addition to all these, the relationship between the variables that are thought to affect the ability to prove and proof can be investigated. Moreover, studies that reveal the increase in students' proof-making skills with the teachers at various education levels allocating time to proof in the lessons will emphasize the importance of proving in mathematics teaching.

Recommendations for Practitioners

In light of the findings of these studies, more sensible suggestions can be made to improve pre-service teachers' knowledge systems and classroom teaching on proof. Another suggestion obtained according to the results of this study is the necessity of emphasizing the proof processes in the course contents to increase the students' pre-service teachers' level of proof. The researchers in this study presented intense content for the proof of the formulas and theorems in the course content. This situation has also improved their students' success towards proof. In this respect, it is essential to emphasize the processes of making proofs in the course contents and to explain the formulas and theorems with scientific relations. In particular, educators who teach trigonometry subjects can achieve more permanent learning by using the properties of the unit circle and similarity theorems. In this context, in order to be able to do advanced mathematics, learning environments and proof activities that enable pre-service teachers to see their deficiencies and mistakes should be included.

The most important limitation of the study is that only analysis 1 and analysis 2 course topics are included in the questions asked to pre-service teachers about proof. Theorems and formulas in the Analysis 1 and 2 courses were taught to the pre-service teachers who participated in the research, and their proofs by the researchers who attended this course. For this reason, different results can be obtained in different participant groups in which different teaching methods are applied. The research is limited to pre-service teachers' questions and answers. Another limitation of the study is that only volunteer pre-service teachers participate in the research. The answers to the research questions may differ according to the pre-service teachers at different grade levels. Therefore, the opinions of pre-service teachers who did not participate in the study or did not want to participate may differ. Thus, the same study can be used with different sample groups.

Limitations of Study

This research is limited to the mathematical proofs answered by the pre-service teachers who participated in the research. In terms of the study group, it is limited to four pre-service teachers who voluntarily participated in the research in the 2021-2022 academic year.

Acknowledgment

Contributions of the Researchers; All authors contributed to the manuscript equally. *Financial Support;* The authors declared that this research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. *Conflict of Interest;* The authors have disclosed no conflict of interest. *Ethical Committee Permissions;* In this study, all rules stated to be followed within the scope of "Higher Education Institutions Scientific Research and Publication Ethics Directive" were followed. None of the actions stated under the title "Actions Against Scientific Research and Publication Ethics", which is the second part of the directive, were not taken. The study was approved by the Izmir Demokrasi University Human Subjects Research Ethics Committee (Date: 8 April 2022, Number: 22/04-03).

References

Almeida, D. (2000). A survey of mathematics undergraduates' interaction with proof: Some implications form mathematics education. *International Journal of Mathematical Education in Science and Technology*, *31*(6), 869-890. https://doi.org/10.1080/00207390050203360

Angelides, M. C., & Agius, H. V. (2002). An interactive multimedia learning environment for VLSI built with Cosmos. *Computers & Education*, *39,* 145-160. https://doi.org/10.1016/S0360-1315(02)00028-3

Arsac, G. (2007). Origin of mathematical proof: History and epistemology. In P. Boero (Ed.), *Theorems in schools: From history, epistemology and cognition to classroom practice* (pp. 27-42). Sense Publishing.

Arslan, Ç. (2007). *İlköğretim öğrencilerinde muhakeme etme ve ispatlama düşüncesinin gelişimi [The development of elementary school students on their reasoning and proof ideas]* (Unpublished doctoral dissertation). Uludağ University, Türkiye.

Balacheff, N. (1998). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216-235). Hodder and Stoughton Publishing.

Bell, A. (1976). A study of pupils' proof-explanations in mathematical situations. *Education Studies in Mathematics 7*(1-2), 23-40. https://www.jstor.org/stable/3481809

Bieda, K. N., Ji, X., Drwencke, J., & Picard, A. (2014). Reasoning-and-proving opportunities in elementary mathematics textbooks. *International Journal of Educational Research*, *64,* 71-80. https://doi.org/10.1016/j.ijer.2013.06.005

Botana, F., & Valcarce, J. L. (2002). A dynamic-symbolic interface for geometric theorem discovery. *Computers & Education*, *38,* 21-35. https://doi.org/10.1016/S0360-1315(01)00089-6

Cai, J., & Howson, G. (2012). Toward an international mathematics curriculum. In: Clements, M., Bishop, A., Keitel, C., Kilpatrick, J., Leung, F. (Eds.) *Third international handbook of mathematics education. Springer international handbooks of education*. Springer. https://doi.org/10.1007/978-1-4614-4684-2_29

Çalışkan, Ç. (2012*). 8. sınıf öğrencilerinin matematik başarılarıyla ispat yapabilme seviyelerinin ilişkilendirilmesi [The interrelations between 8th grade class students' mathematics success and proving levels]* (Unpublished master's thesis). Uludağ University, Türkiye.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras 24,* 17-24.

Demircioğlu, H. (2023). Preservice mathematics teachers' proving skills in an incorrect statement: Sums of triangular numbers. *Pegem Journal of Education and Instruction*, *13*(1), 326-333. https://doi.org/10.47750/pegegog.13.01.36

Doruk, M., & Kaplan, A. (2017). İlköğretim matematik öğretmeni adaylarının analiz alanında yaptıkları ispatların özellikleri *[The characteristics of proofs produced by preservice primary mathematics teachers in calculus]. Mehmet Akif Ersoy University Journal of Education Faculty*, *44,* 467-498. https://doi.org/10.21764/maeuefd.305605

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, *21*(1), 6-13. https://doi.org/10.1007/BF01809605

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, *44*(1-2), 5-23. https://doi.org/10.1023/A:1012737223465

Hanna, G., & Barbeau, E. (2002). What is a proof. *History of Modern Science and Mathematics*, *1*, 36-48.

Harel, G., & Sowder, L. (1998). Types of students justifications. *The Mathematics Teacher*, *91*(8), 670-675. https://doi.org/10.5951/MT.91.8.0670

Hartter, B. J. (1995). *Concept image and concept definition for the topic of the derivative* (Unpublished doctoral dissertation). Illinois State University, USA.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428. https://doi.org/10.2307/749651

Jones, K. (1997). Student teachers' conceptions of mathematical proof. *Mathematics Education Review, 9*, 21-32.

Jones, K. (2000). The student experience of mathematical proof at university level. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 53-60. https://doi.org/10.1080/002073900287381

Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*, 61-88. https://doi.org/10.1023/A:1013838713648

Knuth, E. J., Choppin, J. M., & Bieda, K. N. (2009). Proof: Examples and beyond. *Mathematics Teaching in the Middle School, 15*(4), 206-211. https://doi.org/10.5951/MTMS.15.4.0206

Kotelawala, U. M. (2007). *Exploring teachers' attitudes and beliefs about proving in the mathematics classroom* (Unpublished doctoral dissertation). Columbia University, USA.

Lesseig, K. (2016). Investigating mathematical knowledge for teaching proof in Professional development. *International Journal of Research in Education and Science*, *2*(2), 253-270.

Martin, G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, *20,* 41-51. https://doi.org/10.5951/jresematheduc.20.1.0041

Miyazaki, M. (2000). Levels of proof in lover secondary school mathematics. *Educational Studies in Mathematics, 41*, 47-68. https://doi.org/10.1023/A:1003956532587

Moralı, S., Uğurel, İ., Türnüklü, E., & Yeşildere, S. (2006). Matematik öğretmen adaylarının ispat yapmaya yönelik görüşleri *[The views of the mathematics teachers on proving]. Kastamonu Education Journal*, *14*(1), 147-160.

Morris, A. K. (2002). Mathematical reasoning: Adults' ability to make the inductive-deductive distinction. *Cognition and Instruction*, *20*(1), 79-118. https://doi.org/10.1207/S1532690XCI2001_4

Polat, K., & Akgün, L. (2023). High school students' and their teacher's experiences with visual proofs. *Erzincan University Journal of Education Faculty, 25*(1), 126-136. https://doi.org/10.17556/erziefd.1092716

Quinn, A. L. (2009). Count on number theory to inspire proof. *Mathematics Teacher*, *103*(4), 298-304. https://doi.org/10.5951/MT.103.4.0298

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, *52*(3), 319-325. https://doi.org/10.1023/A:1024360204239

Saeed, R., M. (1996). *An exploratory study of college student's understanding of mathematical proof and the relationship of this understanding to their attitude toward mathematics* (Unpublished doctoral dissertation). Ohio University, USA.

Sarı Uzun, M., & Bülbül, A. (2013). A teaching experiment on development of pre-service mathematics teachers' proving skills. *Education and Science*, *38*(169), 372-390.

Schabel, C. (2005). An instructional model for teaching proof writing in the number theory classroom. *Primus: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *15*(1), 45-59. https://doi.org/10.1080/10511970508984105

Schoenfeld, A. (1994). What do we know about mathematics curricula*? Journal of Mathematical Behavior, 13*(1), 55-80. https://doi.org/10.1016/0732-3123(94)90035-3

Strijbos, J. W., Martens, R. L., & Jochems, W. M. G. (2003). Designing for interaction: Six steps to designing computer-supported group-based learning. *Computers & Education*, *42,* 403-424. https://doi.org/10.1016/j.compedu.2003.10.004

Stylianides, G. J., & Stylianides, A. J. (2023). Preservice teachers' ways of addressing challenges when teaching reasoning-and-proving in their mentor teachers' mathematics classrooms. *In practical theorising in teacher education* (pp. 97-112). Routledge Publishing.

Stylianides, G. J., Stylianides, A. J., & Philippou. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10,* 145-166. https://doi.org/10.1007/s10857-007-9034-z

Tavşancıl, E., & Aslan, A. E. (2001). *Content analysis and application examples for verbal, written and other materials.* Epsilon Publishing.

Urhan, S., & Bülbül, A. (2016). The relationship is between argumentation and mathematical proof processes. *Necatibey Faculty of Education Electronic Journal of Science and Mathematics Education, 10*(1), 351-373. https://doi.org/10.17522/nefefmed.00387

Weber, K (2006). Investigating and teaching the processes used to construct proofs. *Research in Collegiate Mathematics Education, 6,* 197-232.

Weber, K. (2001). Student difficulty in constructing proof: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101-119. https://doi.org/10.1023/A:1015535614355

Weber, K., Maher, C., Powell, A. & Lee, H. S. (2008). Learning opportunities from group discussions: warrants become the objects of debate. *Educational Studies in Mathematics*, *68,* 247-261. https://doi.org/10.1007/s10649-008-9114-8

Wheeler, D. (1990). Aspects of mathematical proof. *Interchange*, *21*(1), 1-5. https://doi.org/10.1007/BF01809604

Yıldırım, A., & Şimşek, H. (2008). *Qualitative research methods in social sciences*. Seçkin Publishing,

Yin, R. K. (2003). *Case study research: Design and methods*. Sage Publishing.